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SENSITIVITY TO GROSS ERRORS OF NEW VARIANTS OF $\mathrm{M_{SPLIT}}$ ESTIMATORS

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Abstract. This study examines the sensitivity of shift estimators between competitive parameters in the $\mathrm{M_{split}}$ estimation method to potential disturbances in observational data caused by gross errors. The essence of parameter estimation in a split functional model of geodetic observations is to determine, based on a single set of measurements, competitive values of the unknown parameter vector. In this way, the approach naturally serves as a method for determining parameter vector displacements, for example in the deformation analysis of geodetic networks. Using a Monte Carlo approach, empirical influence functions were obtained for the classical squared $\mathrm{M_{split}}$ estimators of shift of parameters as well as for new variants of this method, including those based on the concept of a strengthening matrix for measurement results corresponding to a specific competing functional model and the elimination of the reversal-point effect. The empirical analyses were conducted using an example concerning the determination of displacements of controlled points in a leveling control network. The obtained results were compared with those produced using the classical least-squares method.

Keywords: $\mathrm{M_{split}}$ estimation, gross errors, empirical influence function, sensitivity.

1. Introduction

The art of measuring the Earth, as the sciences broadly understood as geodesy are commonly referred to, makes use of various types of measuring instruments intended to obtain results of observed quantities. Due to the application of certain measurement techniques and technologies, as well as imperfections of the measuring equipment itself, geodetic observations may be affected by errors of various kinds, such as systematic errors, deterministic errors, or gross measurement errors, which in the relevant literature are also referred to as outlying observations (outliers). It is well known that geodetic observations are inevitably affected by errors of a random nature. As a result, when measuring a given quantity Y, one may expect that the observed value y of this quantity will differ from its true value. Moreover, when repeating the measurement of the same quantity in a series of observations from y_1 to y_n, one can expect that most of them will be assigned different values. Therefore, the fundamental problem in the processing of geodetic observations is the search for the most probable value of the measured quantity. Since the pioneering publications by Legendre (1805), geodetic observations have been adjusted using the Least Squares (LS) estimation method, in which the optimization criterion assumes minimization of the sum of squares of observational residuals or theoretical observational corrections. It should be noted that the least squares method is also a special variant of the maximum likelihood method (assuming a normal distribution as the probabilistic error model) and belongs to the broad family of M-estimation methods. LS estimation belongs to the group of neutral estimators, meaning that it is characterized by high sensitivity to the presence of outlying observations in the measurement result vector. In addition to neutral estimators, it is also possible to determine weak and robust estimators which, unlike the least squares method, respectively favor or limit the influence of outlying observations in the process of estimating the true values of the quantities being measured.

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Each of the methods listed above operates under the fundamental assumption that the sets of observations are realizations of a single random variable (and, in the case of robust methods, of a single acceptable random variable). In Wiśniewski (2009), it was pointed out that in geodetic measurement practice (as well as in other fields of metrology) situations may arise in which the observation vector is a natural mixture of realizations of at least two random variables. In such a case, the outlying observations in the observation vector are not the result of gross errors in the measurements, but rather of a natural relationship between realizations of two competing random variables. Consequently, the estimation of the positional parameters of these competing random variables may be carried out by applying a parameter estimation method in a split functional model of geodetic observations, referred to as M_{split} estimation (Wiśniewski, 2009). A characteristic feature of this estimation method is the splitting of the classical functional model into two competing models. The fact that two random variables compete for a single observation in M_{split} estimation means that this method naturally enables the determination of an estimator of the shift between the positional parameters of these variables. Direct determination of the shift between competing estimators is possible using a special version of M_{split} estimation, namely Shift-M_{split} estimation (Duchnowski & Wiśniewski, 2011, 2012; Wiśniewski & Zienkiewicz, 2016).

The new estimation method proposed by Professor Zbigniew Wiśniewski is a generalization of classical M-estimation methods and can be extended to an arbitrary number of competing models (Wiśniewski, 2010). In principle, the determination of an appropriate number of competing functional models may be carried out using a modified data-snooping approach (Zienkiewicz, 2020).

The most popular and at the same time the most practical version of the parameter estimation method in a split functional model is squared M_{split} estimation. Such a version of the method may result, among other things, from adopting the normal distribution as the probabilistic error model (Wiśniewski, 2009, 2010). Squared M_{split} estimation has already found a number of practical applications, for example in the processing of observation vectors contaminated with gross errors (Li et al., 2013), in the search for a stable reference datum during the analysis of deformations of geodetic networks (Nowel, 2019; Zienkiewicz, 2022), in the geodetic determination of displacements of monitored points (Wiśniewski et al., 2019; Wyszkowska & Duchnowski, 2020), as well as in fitting appropriate shapes and objects to 3D point clouds acquired using laser scanning technology (see, e.g., Janowski 2018; Wyszkowska & Duchnowski, 2022; Zienkiewicz & Dąbrowski, 2023).

The practical properties of M_{split} estimators have naturally contributed to the development of the theory of this method. Subsequent research on the development of M_{split} estimation has led to the formulation of a classical accuracy analysis of the estimated competing quantities, based on the law of covariance matrix propagation and on the estimation of global and quasi-local variance factors for competing functional models (Wiśniewski & Zienkiewicz, 2021b). Further developments include the estimation of competing parameters in errors-in-variables models (Wiśniewski, 2022), as well as the proposal of a group of robust M_{split} estimators in which the components of the objective function are robust weighting functions commonly used in robust M-estimation (Wyszkowska & Duchnowski, 2022). Nevertheless, from a practical point of view, the most important extensions of squared M_{split} estimation are $M_{split}^{\mathrm{Bind}}$ and $M_{split}^{\Diamond}$ estimation. In the former, the theory of parameter estimation in a split functional model was supplemented with equations of conditional constraints imposed on the competing estimators. In Zienkiewicz (2019), it was demonstrated that the introduction of a specific type of constraints can solve a relatively troublesome problem encountered in practical applications, namely the interchange of M_{split} estimators. It should be noted that the existence of the so-called reversal point (Duchnowski & Wiśniewski, 2017) during the determination of competing estimators may (though does not have to) be problematic from the point of view of interpreting the obtained results – for example, in the geodetic determination of displacements. In turn, the $M_{split}^{\Diamond}$ estimation proposed in (Zienkiewicz & Dąbrowski, 2023) is characterized by the introduction of a matrix that strengthen the assignment of observations to a given competing functional model. It was observed that in certain cases – especially in $M_{split(q)}$ estimation for $q>2$ – a "strong" outlying behavior of observations from the perspective of competing functional models may disturb the results of the estimation of competing parameters. Although this is not a frequent situation, when it does occur it leads to a complete breakdown of the process of determining competing estimators. From a numerical point of view, the introduction of the strengthening matrix aims to limit the influence of "strongly" outlying observations in the cross-weighting process on the estimated competing estimators. In this case, such an observation is unambiguously assigned to a given functional model, thereby eliminating its influence on the estimation of other quantities. Moreover, Zienkiewicz and Dąbrowski (2023) showed that in some cases the strengthening matrix can "suppress" outlying observations. This is a particularly important property, since M_{split} estimators, similarly to LS estimators, exhibit high sensitivity to gross errors in measurements (Wiśniewski & Zienkiewicz, 2021a). Nevertheless, in certain problems M_{split} estimation may be considered in the same context as robust M-estimation (Wiśniewski, 2009; Wiśniewski & Duchnowski, 2017; Nowel, 2019). The robustness properties of classical squared M_{split} estimation, as well as the response of competing estimators to the occurrence of gross errors in observations, were tested using a Monte Carlo approach. For this purpose, performance indicators were determined in the context of gross error detection, along with Empirical Influence Functions (EIFs) that illustrate the impact of a given disturbance in an observation on the values of the estimated competing estimators (Wiśniewski & Zienkiewicz, 2021a).

It is worth noting that Duchnowski and Wiśniewski (2014) also investigated the sensitivity of squared Shift-M_{split} estimators to gross errors present in the observation vector and empirically determined the accuracy of the obtained estimates. It is assumed that the modifications of the optimization criteria of squared M_{split} estimation introduced in Zienkiewicz (2019), Zienkiewicz and Dąbrowski (2023) contribute to an increase in the accuracy of estimating competing estimators and should, within certain ranges, also limit the effect of gross errors on the obtained estimates of the shift between competing parameters. Therefore, in this article the numerical experiments from the study Duchnowski and Wiśniewski (2014) will be repeated, within which the accuracy of the estimated shift determined using the $M_{\mathrm{split}}^{\mathrm{Bind}}$ and $M_{\mathrm{split}}^{\Diamond}$ estimation methods will be evaluated. In addition, the empirical influence functions of these methods will be determined, illustrating how a gross error in an observation affects the values of the computed displacement between the newly proposed variants of squared M_{split} estimators. Numerical examples concerning the processing of measurement results of a leveling control network will be preceded by a brief theoretical introduction to the classical and new variants of competing parameter estimation in a split observation model. The paper concludes with a summary and conclusions drawn on the basis of the conducted research.

2. Theoretical foundations

2.1. Squared M_{split} estimation

The theoretical foundations of M_{split} estimation were formulated under the fundamental assumption that two random variables compete for each observed quantity. Consequently, the classical functional model is split into two competing models with the following linear or linearized form Wiśniewski (2009):

$$\begin{cases} \boldsymbol{\varepsilon}_{(1)} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_{(1)} \\ \boldsymbol{\varepsilon}_{(2)} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_{(2)} \end{cases}, \tag{1}$$

where, $\boldsymbol{y}$ (n × 1) is a vector of measurement results, $\boldsymbol{x}_{(1)}, \boldsymbol{x}_{(2)}$ are competing versions of the parameter $\boldsymbol{x}$ of dimension (r × 1), $\boldsymbol{\varepsilon}_{(1)}$ and $\boldsymbol{\varepsilon}_{(2)}$ are competing versions of the random measurement error vector $\boldsymbol{\varepsilon}$ (n × 1), such that $\boldsymbol{\varepsilon} = -\boldsymbol{v}$, where $\boldsymbol{v}$ is the vector of theoretical observation corrections, whereas $\boldsymbol{A}$ (n × r) is the design matrix. Let us assume that the coefficient matrix is of full column rank, i.e., $rank(\boldsymbol{A}) = r$ and that the observations are mutually independent, with the following split statistical models: $\boldsymbol{C}_{y_{(1)}} = \sigma^2_{0(1)}\boldsymbol{P}^{-1}$ and $\boldsymbol{C}_{y_{(2)}} = \sigma^2_{0(2)}\boldsymbol{P}^{-1}$, where $\boldsymbol{P}$ is the observation weight matrix, whereas $\sigma^2_{0(1)}$ and $\sigma^2_{0(2)}$ are the variance factors corresponding to models (1) and (2), respectively. The sought M_{split} estimators are the quantities that minimize the following optimization criterion Wiśniewski (2009), Zienkiewicz (2022):

$$\varphi\left(\boldsymbol{x}_{(1)}, \boldsymbol{x}_{(2)}\right) = \sum_{i=1}^{n} p_i^2 \varepsilon_{i(1)}^2 \varepsilon_{i(2)}^2 = \min_{\boldsymbol{x}_{(1)}, \boldsymbol{x}_{(2)}}, \tag{2}$$

where, p_i is the weight of the i-th observation. The optimization criterion (2) can be solved, for example, using the method of setting the gradient of the objective function $\varphi\left(\boldsymbol{x}_{(1)}, \boldsymbol{x}_{(2)}\right)$ to zero. In that case, the quantities $\hat{\boldsymbol{x}}_{(1)}$ and $\hat{\boldsymbol{x}}_{(2)}$ are determined on the basis of the following iterative process (for l = 1, 2 and j = 1,…,m):

$$\begin{cases} \boldsymbol{x}_{(1)}^j = (\boldsymbol{A}^T\boldsymbol{P}^2\boldsymbol{W}_{(1)}^{j-1}\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{P}^2\boldsymbol{W}_{(1)}^{j-1}\boldsymbol{y} \\ \boldsymbol{\varepsilon}_{(1)}^j = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_{(1)}^j \\ \boldsymbol{x}_{(2)}^j = (\boldsymbol{A}^T\boldsymbol{P}^2\boldsymbol{W}_{(2)}^{j}\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{P}^2\boldsymbol{W}_{(2)}^{j}\boldsymbol{y} \\ \boldsymbol{\varepsilon}_{(2)}^j = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_{(2)}^j \end{cases}, \tag{3}$$

where,
$\boldsymbol{W}_{(1)} = diag\left(\varepsilon_{1(2)}^2, \ldots, \varepsilon_{n(2)}^2\right)$ and $\boldsymbol{W}_{(2)} = diag\left(\varepsilon_{1(1)}^2, \ldots, \varepsilon_{n(1)}^2\right)$ are cross-weighting matrices. The iterative process may start from randomly selected numerical values; however, most often the initial elements are assumed to be LS estimators (Wiśniewski, 2009). The iterative process is terminated when the quantities $\hat{\boldsymbol{x}}_{(1)} = \boldsymbol{x}_{(1)}^m = \boldsymbol{x}_{(1)}^{m-1}$, $\hat{\boldsymbol{x}}_{(q)} = \boldsymbol{x}_{(q)}^m = \boldsymbol{x}_{(q)}^{m-1}$ when the gradients of the objective function of squared M_{split} estimation are reduced to zero. Based on the estimated competing parameters, the displacement between these values is generally determined in an indirect manner:

$$\hat{\boldsymbol{\Delta}}_{M_{split}} = \hat{\boldsymbol{x}}_{(2)} - \hat{\boldsymbol{x}}_{(1)}. \tag{4}$$

Note that exactly the same result for determining the displacement between competing parameters can be obtained by applying the direct approach to "shift" estimation, i.e., Shift-M_{split} estimation (see, e.g., Wiśniewski & Zienkiewicz, 2016). It should be emphasized, however, that due to the occurrence of the reversal point effect, an interchange of the competing estimators determined in process (3) may take place. In such a case, there is a risk that in certain practical problems – such as the determination of displacements in geodetic control networks – only the absolute value of the shift between these estimators will be correctly determined on the basis of M_{split} estimators.

2.2. $M_{\text{split}}^{\text{Bind}}$ estimation (elimination of the reversal point)

The general idea of this approach is to apply constraint equations that bind the competing parameters (Zienkiewicz, 2019). The search for a conditional minimum of the M_{split} estimation optimization criterion is carried out using the Lagrange multiplier method and may concern, among other things, free network adjustment (Zienkiewicz, 2022) or parameter estimation in errors-in-variables models (Wiśniewski, 2022). In geodetic practice, these constraints may apply to the competing coordinates of certain network points, for example when the distance between these points is determined with high accuracy, or when the constraints between coordinates result from the geometric structure of the network (Dąbrowski et al., 2023). In the present study, the Lagrange method will be applied within squared M_{split} estimation in order to eliminate the reversal point effect in the process of determining displacements of points of a geodetic control network measured in two observation epochs. In this case, the M_{split} estimator of the first competing functional model (see Eq. (1)) will be permanently bind to the least squares estimator $\hat{\boldsymbol{x}}_{LS}$ determined for observations acquired exclusively in the first measurement epoch. Consequently, the conditional objective function of squared M_{split} estimation takes the following form:

$$\varphi_{Lagrange}\left(\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)}\right)= \\ =\sum_{i=1}^{n} p_i^2 \varepsilon_{i(1)}^2 \varepsilon_{i(2)}^2 + c_{(1)} \boldsymbol{\kappa}_{(1)}^T \left(\boldsymbol{x}_{(l)} - \hat{\boldsymbol{x}}_{LS}\right) = \min_{\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)}}, \quad (5)$$

where, $c_{(1)}$ is a coefficient with an arbitrarily selectable value (different from zero), whereas $\boldsymbol{\kappa}_{(1)}^T$ (w × 1) is an vector of Lagrange multipliers corresponding to the constraints imposed on the parameters in model (1). The objective function (5) is minimized by those $M_{\text{split}}^{\text{Bind}}$ estimators that reduce the gradients of this function to zero. In the present paper, due to the limited number of pages available for the description of the research, the scope of the presented theoretical background has been kept to a minimum. The method for determining the quantities $\hat{\boldsymbol{x}}_{(1)}^{Bind} = \hat{\boldsymbol{x}}_{LS}$ and $\hat{\boldsymbol{x}}_{(2)}$ is described in detail in Zienkiewicz (2019). Nevertheless, it should be noted that, based on the obtained $M_{\text{split}}^{\text{Bind}}$ estimators, the displacement between these quantities can also be estimated according to relation (4). Such a displacement will hereafter be denoted as $\hat{\Delta}_{M_{split}}^{Bind}$.

2.3. $M_{\text{split}}^{\Diamond}$ estimation and the strengthening matrix

The theory of M_{split} estimation assumes that the elementary split potential determines the probability of assigning an observation to a given functional model. In practice, such an assignment is performed on the basis of cross-weighting matrices. Each observation is then assigned a certain non-zero value in matrices $\boldsymbol{W}_{(1)}$ and $\boldsymbol{W}_{(2)}$, resulting from the probability of the observation belonging to models (1)–(2), respectively. Therefore, if the observation set contains realizations of a random variable with "extremely" large values compared to other observations, the elementary split potential will nevertheless assign such observations a certain probability of participation in the model to which observations with smaller values are "assigned." In order to eliminate the influence of observations unambiguously assigned to one of the competing models on another model (or models, in the case of $M_{split(q)}$ estimation), Zienkiewicz and Dąbrowski (2023) proposed introducing indicators t_i that strengthen the process of identifying the i-th observation with one of the mutually competing functional models. This parameter takes only two values, namely 1 when there are no grounds for excluding the i-th observation from the computational process, and 0 when a given observation should not participate in the process of determining a given competing estimator. Based on these assumptions, the following optimization criterion can be formulated on the basis of the optimization criterion of squared M_{split} estimation (Zienkiewicz & Dąbrowski, 2023):

$$\varphi_{\Diamond}\left(\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)}\right)=\sum_{i=1}^{n} p_i^2 \varepsilon_{i(1)}^2 \varepsilon_{i(2)}^2 t_i = \min_{\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)}}. \quad (6)$$

The quantities fulfilling the above optimization criterion are referred to as $M_{\text{split}}^{\Diamond}$ estimators. The competing parameters are determined using a diagonal matrix $\boldsymbol{T} = diag\left(t_1,\ldots,t_n\right)$, whose coefficients are determined iteratively and individually for each competing functional model. In the initial phase of the squared M_{split} estimation process, it is necessary to assume $t_i = 1$. Subsequently, in successive iterations ($j_{\Diamond} = 1,\ldots,m_{\Diamond}$), the values of the parameters t_i are determined according to the following conditional relation: (for l = 1, 2):

$$t_{i(l)}^{j_{\Diamond}} = \begin{cases} 1 & \text{for} \quad (\left|\bar{\varepsilon}_{i(l)}\right| \le a) \vee (\hat{\sigma}_{\hat{\varepsilon}_{i(l)}} \le \sigma_{i_{\max}}) \\ 0 & \textit{if} \quad (Z_{(l),\max}^{*} = \left|\bar{\varepsilon}_{i(l)}^{*}\right|) \wedge (\left|\bar{\varepsilon}_{i(l)}^{*}\right| > a) \end{cases}, \quad (7)$$

where, $\bar{\varepsilon}_{i(l)} = \hat{\varepsilon}_{i(l)} / \left(\hat{\sigma}_{0(l)} m_{\hat{\varepsilon}_{i(l)}}\right) = \hat{\varepsilon}_{i(l)} / \hat{\sigma}_{\hat{\varepsilon}_{i(l)}}$ is the standardized residual of an observation corresponding to the l-th functional model, $\hat{\sigma}_{\hat{\varepsilon}_{i(l)}}$ are the competing a posteriori standard deviations of the computed residuals, $\hat{\sigma}_{0(l)}$ are split quasi-local standard deviation factors, which in the present approach assume theoretical values, i.e., $\hat{\sigma}_{0(l)}^2 = 1$. Then, $m_{\hat{\varepsilon}_{i(l)}}$ is determined as the square root of the elements located on the main diagonal of the given variance–covariance matrix $\boldsymbol{C}_{\varepsilon_{(l)}}$ (Wiśniewski & Zienkiewicz, 2021b; Zienkiewicz & Dąbrowski, 2023).

Moreover, in expression (7), the quantity a is a parameter steering the $M_{\text{split}}^{\Diamond}$ estimation process, which is selected arbitrarily (in the present study a = 2.5), whereas $Z_{(l),\max}^{*}$ is the absolute value of the standardized residual of the observation with the largest value in the l-th functional model. Moreover, the quantities $\left|\bar{\varepsilon}_{i(l)}^{*}\right|$ and $Z_{(l),\max}^{*}$ refer to the absolute values of the standardized quantities $\varepsilon_{i(l)}$, whose standard deviations are $\hat{\sigma}_{\hat{\varepsilon}_{i(l)}} > \sigma_{i_{\max}}$. During the conducted analyses, it was assumed that the threshold value is $\sigma_{i_{\max}} = \sqrt{0.02}\sigma_i$.

Eq. (7) should be interpreted as follows: after performing the classical M_{split} estimation process, the a posteriori standardized observational residuals are subjected to analysis. If the absolute value of $\left|\bar{\varepsilon}_{i(l)}\right|$ does not exceed the critical value a, then the parameter t corresponding to the i-th observation in the l-th functional model takes the value equal to 1. The parameter $t_{i(l)}^{j_{\Diamond}}$ is also equal to one in the case when the i-th observation is assigned to the l-th model without error, i.e., when $\hat{\sigma}_{\hat{\varepsilon}_{i(l)}} \le \sigma_{i_{\max}}$. However, if in the l-th considered competing model there are quantities $\left|\bar{\varepsilon}_{i(l)}^{*}\right|$ exceeding the threshold value, then for the observation with the largest absolute value of the a posteriori standardized random error in that model, the parameter t is assigned a value of zero. Such a procedure is performed independently for each of the competing functional models. In this way, specific forms of the matrices $\boldsymbol{T}_{(1)}$ and $\boldsymbol{T}_{(2)}$ are obtained at iteration $j_{\Diamond}$. The $M_{\text{split}}^{\Diamond}$ parameter estimation procedure is repeated until all quantities $\left|\bar{\varepsilon}_{i(l)}\right|$ are smaller than the threshold value a (this applies to the competing versions of standardized observation corrections for which $t_{i(l)} = 1$ holds). By differencing the $M_{\text{split}}^{\Diamond}$ estimators according to relation (4), the parameter shift $\hat{\Delta}_{M_{split}}^{\Diamond}$ is obtained.

3. Numerical example

As mentioned earlier, the sensitivity of the new versions of squared M_{split} estimators will be determined by repeating selected numerical experiments presented in (Duchnowski & Wiśniewski, 2014). The empirical analyses will be supported by Monte Carlo simulation of observation data. The computations will concern the leveling control network shown in Figure 1.

The control network consists of two fixed points P1 and P2 with theoretical heights $H_{P1} = H_{P2} = 0.000$ m. In addition, the network structure is supplemented by three monitored points A, B and C with theoretical height $H_A = H_B = H_C = 1.000$ m. It was assumed that eight height differences $h_1,\ldots,h_8$ between the points were measured in two observation epochs. Observations of the sixteen height differences were simulated by adding random errors to the theoretical height differences of the network points, generated from a normal distribution with an expected value of 0 and a standard deviation of 0.001 m.

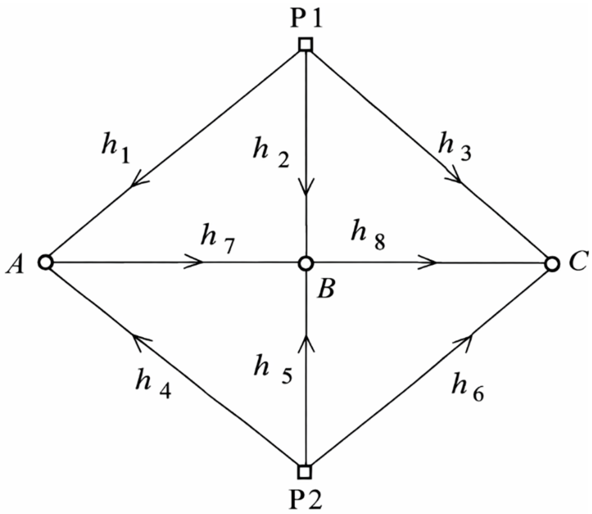

Figure 1. Sketch of the leveling network

Moreover, the height differences in the second measurement epoch were generated for four variants of changes in the theoretical heights of the monitored points, namely:

Variant I: $H_C = 1.010$ m;

Variant II: $H_C = 1.010$ m and $H_B = 1.030$ m;

Variant III: $H_C = 1.010$ m, $H_B = 1.005$ m and $H_C = 0.995$ m;

Variant IV: $H_C = 1.010$ m, $H_B = 1.030$ m and $H_C = 0.990$ m.

The observation sets for both measurement epochs were generated 50 000 times for each variant. Based on the simulated data, displacements of the control network points were determined using squared M_{split}, $M_{\text{split}}^{\text{Bind}}$ and $M_{\text{split}}^{\Diamond}$ estimation. The obtained results were compared with the displacement estimates $\hat{\Delta}_{LS}$ derived using the least squares method. It should be noted that in LS estimation, the heights of the control network points are estimated separately for each of the considered measurement epochs. Additionally, in order to compare the accuracy of the determined displacements for each considered estimation method, the Root-Mean-Square Deviation (RMSD) of the displaced point was computed according to the mathematical formula given in (Duchnowski & Wiśniewski, 2014):

$$RMSD = \sqrt{\sum_{u=1}^{k} (\Delta - \hat{\Delta}_u)^2 / k}, \tag{8}$$

where, Δ is the theoretical value of the displacement of a given point, whereas $\hat{\Delta}_u$ is the estimator of this displacement in the u-th data simulation (out of k possible simulations). The results of the estimation of displacements of the leveling control network points, together with the RMSD error, are presented in Table 1.

Table 1. Results of displacement estimation

Variant	Simulation results (mm)							
	LSE		M_{split}		M_{split}^{Bind}		$M_{split}^{\Diamond}$	
	Mean	RMSD	Mean	RMSD	Mean	RMSD	Mean	RMSD
I	10.0	0.87	10.1	1.00	10,0	0.95	10.1	1.03
	0.0	0.78	0.1	1.70	0.1	1.30	0.1	1.71
	0.0	0.86	0.0	1.93	0.0	1.23	0.1	1.94
II	10.0	0.86	10.0	1.12	10.0	1.01	10.1	1.23
	30.0	0.78	30.0	0.94	30.0	0.86	30.0	0.93
	0.0	0.86	0.0	1.68	0.0	1.34	0.0	1.7
III	10.0	0.87	10.0	1.14	10.0	1.00	10.0	1.19
	30.0	0.78	30.0	0.88	30.0	0.83	30.0	0.93
	−10.0	0.87	−10.0	1.40	−10.0	1.16	−10.0	1.43
IV	10.0	0.86	10.0	1.48	10.0	1.20	10.0	1.50
	5.0	0.78	5.0	0.92	5.0	0.84	5.0	0.95
	−5.0	0.97	−5.0	1.75	−5.0	0.89	−5.0	0.98

The obtained results clearly indicate that each of the analyzed methods led to a correct assessment of the "shift" between the considered parameters. It should be noted, however, that the values of the competing M_{split} estimators were "artificially" assigned to the appropriate measurement epoch. In practical applications of this method, it is possible to automatically determine only the absolute value $\hat{\Delta}_{M_{split}}$ correctly. This problem does not occur in the case of the quantity $\hat{\Delta}^{Bind}_{M_{split}}$. The introduced constraints protect against the interchange of competing estimators without adversely affecting the values of the estimated shift. Moreover, the displacement determined using M_{split}^{Bind} estimation is characterized by a lower RMSD error value than that obtained with classical M_{split} estimation. Consequently, the accuracy of the determined displacements approaches that achieved using LS estimation. The introduction of the strengthening matrix in the estimation process had practically no effect on the results of determining competing estimators or on the RMSD errors. This is due to the fact that the observation vector contained realizations of exactly the same number of random variables as the number of competing models assumed. In the case of M_{split} estimation, the strengthening matrix begins to play a role when, for example, gross errors appear in the observation vector Zienkiewicz and Dąbrowski (2023), although its primary purpose is to limit "highly" outlying observations in squared $M_{split(q)}$ estimation.

In the next step of the empirical analyses, we therefore examine the effect that gross errors in observations have on the values of the determined displacements. In paper Duchnowski and Wiśniewski (2014), an empirical influence function was defined to describe the sensitivity of the estimators $\hat{\Delta}$ to disturbances introduced into the observation vector:

$$EIF(x) = T_n(\boldsymbol{y}_1, \boldsymbol{y}_2 + \boldsymbol{g}), \tag{9}$$

where, T_n is the estimator under investigation, $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are the observation vectors in epochs 1 and 2, respectively, whereas $\boldsymbol{g} = \left[0 \ldots x \ldots 0\right]^T$ is a vector indicating the position x at which a gross error will be added to one of the observations of the second measurement epoch. In the present study, following the approach adopted by Professor R. Duchnowski and Professor Z. Wiśniewski, it was assumed that the gross error would be added to observation h_2 in Variant I, and subsequently the displacements would be determined using LS, M_{split}, M_{split}^{Bind} and

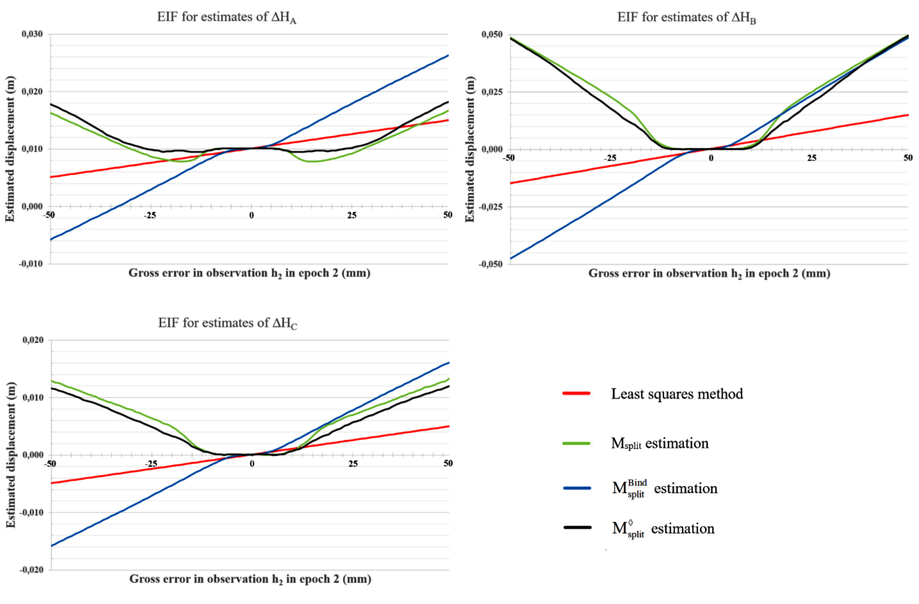

Figure 2. Empirical influence function of the displacement of control network points (Variant I)

$M^{\lozenge}_{split}$ estimation. The value of the disturbance x will increase by 1 mm within the interval -50 mm, 50 mm. It is assumed that the vectors $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ will be generated 10,000 times, and on this basis the mean values of the estimator T_n will be determined for each variant of the gross error in observation h_2. The courses of the empirical influence functions related to the estimation of the displacements of points A, B and C are shown in Figure 2.

The determined EIFs show, as expected, that LS estimators and all variants of M_{split} estimators for the displacements of control network points exhibit high sensitivity to gross errors. It should be noted, however, that the results of classical M_{split} and $M^{\lozenge}_{split}$ estimation show higher sensitivity than LSE to so-called "large" outliers, whereas in the case of "small gross errors" the EIFs indicate lower sensitivity than that observed for LS estimation. This is particularly important, as it is well known that large gross errors are relatively easy to detect using statistical tests. The determined empirical influence functions and RMSD errors confirm that the new variants of competing parameter estimation in a split functional model provide more robust and more reliable results than classical squared M_{split} estimators.

4. Conclusions

The study analyzed the sensitivity of classical and modified M_{split} estimators to the presence of gross errors in geodetic observations using Monte Carlo simulations and empirical influence functions. The obtained results confirmed that the application of conditional constraints efficiently eliminates the reversal point effect while simultaneously improving the accuracy of the determined displacements compared to classical quadratic M_{split} estimation. An analysis of RMSD error values showed that conditional estimation leads to results close to those obtained using the least squares method. Empirical influence functions indicate a greater sensitivity of LS estimation to small gross errors compared to the considered variants of M_{split} estimation. For large disturbance values, all methods exhibit similar behavior, although estimators of competing parameters show slightly greater susceptibility to this type of perturbation.

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